# Sec 5.5 Nonhomog Eqs (method of undeter. ~~m~~ coeffs part 2)

"Problem Case" $y'' - y' - 2y = e^{2x}$

- $y_c = c_1 e^{-x} + c_2 \boxed{e^{2x}}$
- since $f(x) = e^{2x} \Rightarrow$ trial $y_p = Ae^{2x}$

$$y_p' = 2Ae^{2x}$$
$$y_p'' = 4Ae^{2x}$$

plug in to check $y_p'' - y_p' - 2y_p = e^{2x}$ ?

$$4Ae^{2x} - 2Ae^{2x} - 2Ae^{2x} \stackrel{?}{=} e^{2x}$$
$$(4A - 4A)e^{2x} = e^{2x}$$
$$0 = e^{2x}$$

... :( trouble.

What went wrong?

- $y_p = Ae^{2x}$ is multiple of one of the "basis solutions" in $y_c$, which by defin must satisfy $y'' - y' - 2y = 0$.

so trying $y_p$, we got $(Ae^{2x})'' - (Ae^{2x})' - 2(Ae^{2x})$

But this is $A\Big(\underbrace{(e^{2x})'' - (e^{2x})' - 2(e^{2x})}_{\text{this } = 0}\Big)$

(b/c $e^{2x}$ satisfies $y''-y'-2y=0$)

- in summary, problem is that $y_p$ has a piece that is not linearly indep. from the "basis functions" "$y_1, y_2$" that make up $y_c$

$$y_c = c_1 \underbrace{\boxed{e^{-x}}}_{y_1} + c_2 \underbrace{\boxed{e^{2x}}}_{y_2}$$

problem trialing $y_p = Ae^{2x} = Ay_2$

So to fix this dependence, we change our trial function $y_p = Ae^{2x} \longrightarrow \underbrace{y_p = Axe^{2x}}_{\text{new trial}}$

note that now $y_p$ is linearly indep from $y_1 = e^{-x}$, $y_2 = e^{2x}$.

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So now $y_p = Axe^{2x}$

$$y_p' = Ae^{2x} + 2Axe^{2x} = A(1+2x)e^{2x}$$

$$y_p'' = 2Ae^{2x} + A(1+2x)\cdot 2e^{2x}$$

$$= 2A(2+2x)e^{2x} = 4A(1+1x)e^{2x}$$

plug in to $y''-y'-2y = e^{2x}$

$$4A(1+x)e^{2x} - A(1+2x)e^{2x} - 2(Ax)e^{2x} = e^{2x}$$

$$e^{2x}(4A-A) + xe^{2x}\cancel{(4A-2A-2A)} = e^{2x}$$

$$3Ae^{2x} = e^{2x} \Longrightarrow \boxed{A = 1/3}$$

So $y_p = Axe^{2x} = \frac{1}{3}xe^{2x}$

Step 3 combine $y_c$, $y_p$

General solution for $y'' - y' - 2y = e^{2x}$

is $y = y_c + y_p$

$$\boxed{y = c_1 e^{-x} + c_2 e^{2x} + \frac{1}{3}xe^{2x}}$$

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recap: $y'' - y' - 2y = e^{2x}$

- $y_c = c_1 \boxed{e^{-x}} + c_2 \boxed{e^{2x}} = c_1 y_1 + c_2 y_2$
- $f(x) = e^{2x}$
- "original trial" $y_p = A\boxed{e^{2x}}$ had "duplication" with (part of) $y_c$, specifically, $y_p$ was not lin. ind. from $y_2 = e^{2x}$. ("duplication")
- To "remove duplication" we changed ~~$y_p$~~

  $$y_p = Ae^{2x} \longrightarrow y_p = A\underline{\underline{x}}e^{2x}$$

  and then used this new $y_p$ as trial.
- We solved for A ..., found $y_p = \frac{1}{3}xe^{2x}$
- Finally we combined $y = y_c + y_p$

  $$= c_1 e^{-x} + c_2 e^{2x} + \frac{1}{3}xe^{2x}.$$

# Hyperbolic trig functions

$\cosh(x) \overset{DEF}{=} \dfrac{e^{x}+e^{-x}}{2}$ $\qquad$ $\cos(x) = \dfrac{e^{ix}+e^{-ix}}{2}$

$\sinh(x) \overset{DEF}{=} \dfrac{e^{x}-e^{-x}}{2}$ $\qquad$ $\sin(x) = \dfrac{e^{ix}-e^{-ix}}{2i}$

"vice versa" have
- $e^{x} = \cosh(x) + \sinh(x)$
- $e^{-x} = \cosh(x) - \sinh(x)$

$$\boxed{\begin{aligned} \frac{d}{dx}\cosh(x) &= \sinh(x) \\ \frac{d}{dx}\sinh(x) &= \cosh(x) \end{aligned}}$$

$\longrightarrow$ cosh, sinh are two lin indep solutions of

$y'' = y$

<u>Ex</u>: $y'' - y = \cosh(x)$

<u>Step 1</u> : $y_c$ : char eq $r^2 - 1 = 0 \Rightarrow r = \pm 1$,

so gen. solution is $\underline{y_c = c_1 \boxed{e^{-x}} + c_2 \boxed{e^{x}}}$

<u>Step 2 (Option 1)</u> : Changing cosh, sinh to exponentials

$f(x) = \cosh x = \frac{1}{2}e^{-x} + \frac{1}{2}e^{x}$

so think trial $y_p = A\underline{e^{-x}} + B\underline{e^{x}}$

need to check for potential duplication with $y_c$.

- Both $e^{-x}$ and $e^{x}$ have duplication with $y_c$

so we change $e^{-x} \to xe^{-x}$, $e^{x} \to xe^{x}$ for $y_p$

So new trial $y_p = A\underbrace{xe^{-x}} + B\underbrace{xe^{x}}$

no duplication with $y_c$ now.

$$y_p' = Ae^{-x} - Axe^{-x} + Be^{x} + Bxe^{x}$$
$$= A(1-x)e^{-x} + B(1+x)e^{x}$$

$$y_p'' = -Ae^{-x} - A(1-x)e^{-x} + Be^{x} + B(1+x)e^{x}$$
$$(-2A + Ax)e^{-x}$$
$$A(-2+x)e^{-x} + B(2+x)e^{x}$$

Orig ODE was $y'' - y = \cosh(x)$

$$y'' - y = \frac{1}{2}e^{-x} + \frac{1}{2}e^{x}$$

plug in $y_p$:

$$A(-2+x)e^{-x} + B(2+x)e^{x}$$
$$- Axe^{-x} - Bxe^{x}$$

$$e^{-x}(-2A) + xe^{-x}(A-A) + e^{x}(2B) + xe^{x}(B-B)$$

$$-2Ae^{-x} + 2Be^{x} \overset{\text{(want)}}{=} \frac{1}{2}e^{-x} + \frac{1}{2}e^{x}$$

$$\Rightarrow \begin{cases} -2A = 1/2 & \Rightarrow A = -1/4 \\ 2B = 1/2 & \Rightarrow B = 1/4 \end{cases}$$

$$\boxed{y_p = -\frac{1}{4}e^{-x} + \frac{1}{4}e^{x}} = \frac{1}{2}\left(\frac{e^{x} - e^{-x}}{2}\right)$$
$$\boxed{= \frac{1}{2}\sinh(x)}$$

see notes for option 2

$\underline{Ex}$: $y^{(5)} + 2y^{(4)} - y = 3$, <u>only</u> find $y_p$

$f(x) = 3$, so we trial $y_p = A$; as long as there is no duplication with $y_c$.

what is $y_c$?

Start with ch. eq. $r^5 + 2r^4 - 1 = 0$.

This seems hard to fully factor, but we only want to check that $y_c$ doesn't have a piece/basis function that is $c_1 \cdot 1$ $(c_1 e^{0x})$;

so, is $r=0$ a root of char eq? *

No, b/c $0 + 0 - 1 \neq 0$, so $r \neq 0$ is <u>not a root</u>.

$$y_c = c_1 \overset{e^{r_1 x}}{y_1} + c_2 \overset{e^{r_2 x}}{y_2} + c_3 \overset{\cdots}{y_3} + c_4 y_4 + c_5 \overset{\cdots}{y_5}$$

wanted to check if any $y_1, \cdots, y_5$ could be part of an "$r=0$ piece", meaning one of them would be $= 1$ (const funct) $= e^{0x}$

so original trial $y_p = A \cdot 1$ is fine.

plug $y_p$ into $y_p^{(5)} + 2y_p^{(4)} - y_p = 3$

$$\rightarrow \quad 0 + 2 \cdot 0 - A = 3$$

$$\Rightarrow A = -3$$

so $y_p = -3$ is a particular solution

Ex: In solving the nonhomog ODE

$$y^{(5)} - 6y^{(4)} + 9y^{(3)} = x + 2e^{3x},$$

what is the appropriate form of $y_p$?

Step 1 $y_c = ?$ char. eq $r^5 - 6r^4 + 9r^3 = 0$

$$r^3(r^2 - 6r + 9) = 0$$

$$r^3(r-3)^2 = 0$$

roots are $r = 0$ (mult. 3)

$r = 3$ (mult. 2)

gen solution for assoc. homog. eq ($y_c$)

$$y_c = c_1 \underbrace{1}_{y_1} + c_2 \underbrace{x}_{y_2} + c_3 \underbrace{x^2}_{y_3} + c_4 \underbrace{e^{3x}}_{y_4} + c_5 \underbrace{xe^{3x}}_{y_5}$$

Step 2 trial $y_p$: $f(x) = x + 2e^{3x}$

initially: try $y_p = (A + Bx) + Ce^{3x}$

check for duplication w/ $y_c$.

- $A + Bx$ has duplication with $y_c$ (spec. $y_1, y_2$)
  - $\Rightarrow$ change this to $Ax + Bx^2$
  - check again. Still duplication w/ $y_c$ ($y_2, y_3$)
  - $\Rightarrow$ change to $Ax^2 + Bx^3$ ... still dup.
  - $\Rightarrow$ change to $Ax^3 + Bx^4$, now no duplication ✓
- $Ce^{3x}$ piece has dupl. with $y_c$ (spec. $y_4$)
  - $\Rightarrow$ change to $Cxe^{3x}$ ... still duplication!
  - $\Rightarrow$ change to $Cx^2e^{3x}$ ... no duplication.

In total, trial solution $y_p = Ax^3 + Bx^4 + Cx^2e^{3x}$.